McGILL UNIVERSITY

FACULTY OF SCIENCE

FINAL EXAMINATION

MATHEMATICS MATH-317A

ELEMENTARY NUMERICAL ANALYSIS

Examiner: Professor N. Nigam | Date:Monday, December 16, 2002

Associate Examiner: Professor P. Tupper | Time: 2:00 P.M. - 5:00 P.M.

INSTRUCTIONS

Only non-programmable, non-graphing calculators are permitted.
This is a closed book, closed notes exam. Answer all questions.
This exam comprises the cover and 7 problems. All problems are compulsory.

Question 1)
a) (2 marks) Precisely define **rate of convergence** of a sequence.

b)(2 marks) Precisely define **order of convergence** of a sequence.

c)(2 marks) Define what is meant by **degree of accuracy** of a quadrature rule.

d) (4 marks) Suppose the difference method

$$w_0 = \alpha, \qquad w_{i+1} = w_i + h\Phi(t_i, w_i), \qquad i = 0, ...N-1$$

is used to approximate solutions of the initial value problem

$$\frac{dy}{dt} = f(t,y), \qquad t \in [a,b], \qquad y(a) = \alpha.$$

Define the **local truncation error** of this difference method.

Question 2)
a) (3 points) The goal of this exercise is to compute the root $p = -5$ of

$$f(x) = x^2 + 10x + 25 = 0.$$

To do this, Newton's method was implemented. Given below are $|p_n - p|$ for $n = 0, 1, 2, ..., 5$. The starting point was $p_0 = 0$.

n	$\|p_n - p\|$
0	5
1	2.5
2	1.25
3	0.625
4	0.3125
5	0.15625

- What is the apparent rate of convergence, and how does it compare with the usual rate of convergence? Briefly explain your observations.
- Write down the formula for Newton's method in this case, and find the first 2 iterates p_1, p_2.
- Would you recommend the Bisection method or the Method of False Position for this problem? Briefly justify your answer.

b) (5 points) Derive the constants x_0, x_1, c_1 so that the quadrature formula

$$\int_0^1 f(x)dx = \frac{1}{2}f(x_0) + c_1 f(x_1)$$

has highest possible degree of accuracy.

Question 3)
a) (3 points)

- What is the key difference between Lagrange and Hermite interpolants?
- A natural cubic spline S on $[0, 2]$ has the formula

$$S(x) = \begin{cases} S_0(x) &= 1 + 2x - x^3, & \text{if } 0 \le x < 1 \\ S_1(x) &= 2 + b(x-1) + c(x-1)^2 + d(x-1)^3, & \text{if } 1 \le x \le 2 \end{cases}$$

Find b, c, d.

b)(4 points) We can approximate $f'(x_0)$ by

$$f'(x_0) = N_1(h) - \frac{h^2}{6} f'''(x_0) - \frac{h^4}{120} f^{(5)}(x_0) - \ldots.$$

where $N_1(h) = \frac{f(x_0+h)-f(x_0-h)}{2h}$.

- What is the order of approximation of $N_1(h)$ to $f'(x_0)$?
- Use Richardson's extrapolation to find $N_2(h)$, an $O(h^4)$ approximation.

Question 4)
a)(10 points) Use the **linear shooting method** to approximate the solution $y(\frac{\pi}{4})$ of the boundary value problem

$$y'' = y' + 2y + \cos x, \qquad 0 \le x \le \frac{\pi}{2}, \qquad y(0) = -0.3, \quad y(\frac{\pi}{2}) = -0.1.$$

Use $h = \frac{\pi}{4}$, and the Forward Euler method.

Question 5)(10 points) Find the first two iterates of the Jacobi method for the following linear system

$$\begin{aligned} 3x_1 - x_2 + x_3 &= 1 \\ 3x_1 + 6x_2 + 2x_3 &= 0 \\ 3x_1 + 3x_2 + 7x_3 &= 4 \end{aligned}$$

using an initial guess of $(0, 0, 0)$.

Question 6)(20 points) Derive a finite-difference method (first order **backward** difference in time and second-order centered difference in space) to solve the diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = \alpha^2 \frac{\partial^2 u(x,t)}{\partial x^2}, \qquad 0 \le x \le L,\ t \ge 0,$$

subject to the boundary conditions

$$u(0,t) = u(L,t) = 0, \qquad t \ge 0$$

and the initial condition

$$u(x,0) = f(x), \qquad 0 \le x \le L.$$

Here α is a real constant. Define the space-step to be $\Delta x = \frac{L}{m+1}$ where m is a positive integer, and a time-step $\Delta t > 0$. Use the Taylor series to determine the local truncation error of your finite difference method. Write the finite-difference equations in matrix form, making sure you implement the boundary and initial conditions. Is this a stable method?

Question 7)(20 points) Consider the initial value problem

$$y' = f(t,y) = -\lambda y, \qquad 0 \le t \le T, \quad y(0) = \alpha, \quad \lambda > 0.$$

- Suppose you approximate the solution $y(t)$ using the Forward Euler method with time-step h

  $$w_0 = \alpha, \qquad w_{i+1} = w_i + hf(t_i, w_i), \qquad i = 0, ...N-1$$

  Prove that $w_{i+1} = (1-h\lambda)^{j+1}\alpha, i = 0, 1, ...N-1$. Under what condition on h is the method stable? What is the local truncation error?

- Suppose you approximate the solution $y(t)$ using the Backward Euler method with time-step h

  $$w_0 = \alpha, \qquad w_{i+1} = w_i + hf(t_{i+1}, w_{i+1}), \qquad i = 0, ...N-1$$

  Prove that

  $$w_{i+1} = \frac{\alpha}{(1+h\lambda)^{j+1}}, i = 0, 1, ...N-1.$$

  Under what condition on h is the method stable? What is the local truncation error?

- Under what circumstances would you advise the use of Backward Euler? What is a major disadvantage with this method when applied to general IVP?